SUMUDU TRANSFORM WITH HOMOTOPY PERTURBATION METHOD FOR SOLVING LANE-EMDEN TYPE SINGULAR INITIAL VALUE PROBLEMS

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Abstract: This paper studied Sumudu Transform coupled with Homotopy Perturbation Method for solving Lane–Emden equations as singular initial value problems. Sumudu Transform with Homotopy Perturbation Method was applied to obtain the exact solutions of the linear and nonlinear singular initial value problems. Comparing this method with some other known method, all the problems considered proved that the Sumudu Transform Homotopy Perturbation Method are very powerful and highly effective in solving both linear and nonlinear singular initial value problems. Three examples were considered to demonstrate the applicability and effectiveness of this method.

Keywords: Sumudu transform, Homotopy perturbation method, Lane-Emden, initial value problems

Introduction

The majority of the physical models are nonlinear in nature and several mathematical techniques have been developed by various researchers to handle the nonlinearity of such problems. Actually, the Sumudu Transform (ST) with Homotopy Perturbation Method (HPM) is one of the few handful techniques that can be employed to systems with periodic or discontinuous driving sources.

Lane-Emden type singular initial value problems has several applications in Mathematics and Astrophysics (Adomian and Rach, 1992). The Lane-Emden type differential problem of the form : (Chandrasekhar, 1967)

$$y'' + \frac{2}{x}y' + \tau(x,y) = \psi(x), 0 \le x \le 1 \qquad 1$$

Subject to the initial conditions

$$y(0) = A, \qquad y'(0) = B \qquad 2$$

Where $\tau(x,y)$ is a continuous real valued function.

The numerical solution of Lane-Emden of problem is quite challenging due to the behavior of singular point at origin. Several methods have been applied by different Researchers to obtain the approximate solutions to Lane-Emden problems. Ramos (2008) obtained a series approach to the Lane-Emden equations and compared the results obtained with He's Homotopy perturbation method. Olubanwo *et al.* (2015) obtained solutions of second order nonlinear singular initial problems by Modified Laplace Decomposition Method. Homotopy perturbation method has been greatly employed to obtain solutions to singular initial value problems. Arioel (2007), El-Mistikawy (2009) obtained an approximate solutions of the Lane-Emden equations by using Homotopy Perturbation Method. Homotopy Perturbation Method was developed by He (1999). He (2003) proposed a combination of Homotopy Perturbation Method with other techniques for nonlinear problems. This has been applied to obtain solution of linear and nonlinear differential problems. Olayiwola and Adegoke (2019) combined Homotopy Perturbation method with Laplace Transform to solve singular initial value problems. Sumudu Transform Method was first developed by Watugala (1993). Watugala (1998) apply Sumudu transform to solve differential equations and control engineering problems. Applying Homotopy Perturbation method with Sumudu Transform (Ziane *et al.*, 2015) is a powerful solutions of both linear and nonlinear Lane-Emden type differential equations which provides a better result comparing with the exact solutions. In this paper, Sumudu Transform with Homotopy Perturbation Method is utilized to solve the Lane-Emden type of differential equations

Sumudu transform with Homotopy perturbation method

Here, we discuss the application of the ST-HPM for the solution of the Lame Emden equation:

$$y'' + \frac{2}{x}y' + \tau(x,y) = \psi(x), 0 \le x \le 1$$

Subject to the initial $y(x) = A,\ y'(x) = B$

Multiplying through by x

$$xy'' + 2y' + x\tau(x,y) = x\psi(x), \qquad 3$$

Taking the Sumudu transform

$$S[xy''] + 2S[y'] + S[x\tau(x,y)] = S[x\psi(x)],$$

$$S[xy''] + 2S[y'] = S[x\psi(x) - x\tau(x,y)],$$

Consider the following second order Lame Emden type non homogenous with initial condition.

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \tau(x,y) = \psi(x), 0 \le x \le 1$$

$$y(0) = A, \qquad y'(0) = B \qquad 4$$

Where $\tau(x,y)$is a real function, $\psi(x)$ is a known function and A and B are constants multiplying eqn (4) by x and applying the Sumudu Transform, we obtain

$$S[x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + x\tau(x,y) = x\psi(x)] \qquad 5$$

Using the linearity properties of the Sumudu Transform, we get

$$\frac{dy}{du} - \frac{Y(u)}{u} + 2Y(u) - \frac{y(u)}{u} + S[x\tau(x,y)] = S[x\psi(x,y)] \qquad 6$$

Equation (6) can be written in the form

$$\frac{d}{du}(UY(u)) - y(u) + uS[x\tau(x,y)] = uS[x\psi(x,y)] \qquad 7$$

Integrating equations (7) yields

$$S[y(x)] = \frac{1}{u}\int_0^u y(u)du + \frac{1}{u}\int_0^u u\,S[x\tau(x,y)]du = \frac{1}{u}\int_0^u u\,S[x\psi(x,y)du] \qquad 8$$

$\tau(x,y)$ can be decomposed into two part

$$\tau(x,y) = K(y(x)) + M[y(x)] \qquad 9$$

Where $K[y(x)]$ and $M[y(x)]$ represent the linear and the nonlinear terms respectively, the concept of the Homotopy perturbation method and the He's polynomial can be employed to solve equation (4) and to address the nonlinear term $[y(x)]$. We defined the solution $y(x)$ by infinite series.

$$y(x) = \sum_{n=0}^{\infty} p^n y_n(x) \qquad 10$$

Where the terms $y_n(x)$ are calculated repeatedly.The nonlinear $\tau(x,y)$can be given as;

$$M[y(x)] = \sum_{n=0}^{\infty} p^n H_n(y) \qquad 11$$

Where $M(y)$ is a nonlinear term and $H_n(y)$ is the He's polynomial.

$H_n(y_o, y_1, y_2, \ldots y_n) = \frac{1}{n!}\frac{\delta^n}{\delta p^n}[M(\sum_{n=0}^{\infty} p^1 y_1)]_{p=0}$ n=0,1, …

Substituting equation (10) and (11) into (8) we obtain

$$\int [\sum_{n=0}^{\infty} p^n y_n(x)] - \frac{1}{u}\int_0^u y(0)du + \frac{1}{u}\int_0^u u\sum_{n=0}^{\infty}(\int [H(\sum_{n=0}^{\infty} p^n y_n(x)) + x\sum_{n=0}^{\infty} p^n H_n(y)]du) = \frac{1}{u}\int_0^u [x\psi(x)] \quad 12$$

This is the combination of the Sumudu transform and Homotopy Perturbation Method (ST-HPM) using He's polynomial comparing the coefficients of the corresponding power of P, we obtain the following recursive relation.

$$S[y_0(x)] = \frac{1}{u}\int_0^u y(u)du + \frac{1}{u}\int_0^u u\, S[x\psi(x)]du \quad 13$$

$$S[y_{n+1}(x)] = \frac{1}{u}\int_0^u u(s[xk(y_n(n)) + xH_n(t)])du \quad 14$$

Taking the inverse Sumudu Transform of equations (13) and (14) yields

$$y_0(t) = S^{-1}(\frac{1}{u}\int_0^u y(u)du + \frac{1}{u}\int_0^u u\, S[x\psi(x)]du) = H(t) \quad 15$$

$$y_{n+1}(t) = S^{-1}(\frac{1}{u}\int_0^u u(S[x(y_n(t)) + xH_n(n)])du \quad 16$$

$$H(t) = H_0(t) + H_1(t) \quad 17$$

$$y_0(t) = H_0(t)$$

$$y_1(t) = H_1(t) + S^{-1}(-\frac{1}{u}\int_0^u u(S[xk(y_n(x)) + xH_n(x)])dun \quad 18$$

$$y_{n+1}(t) = S^{-1}(-\frac{1}{u}\int_0^u u(S[xk(y_n(x)) + xH_n(x)])du$$

The solution highly depends on the choice of $H_0(t)$

Results and Discussion

This section demonstrates the effectiveness and applicability of the Sumudu transform and Homotopy Perturbation to solve Lane-Emden type singular initial value problems. The following examples were solved to demonstrate the applicability of the method.

Example 1

Consider the singular homogenous Lame Emden type equation (Eltayeb , Kilicman and Bachar, 2015).

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} - 2(2x^2+3)y = 0 \quad 19$$

With initial conditions

$$y(0) = 1 \;, \quad y'(0) = 1 \quad 20$$

Multiplying equation (19) by x and taking the Sumudu Transform, we have

$$\int [x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2x(2x^2+3)y] = 0 \quad 21$$

Using the initial properties of Sumudu Transform.

$$\frac{dY}{du} - \frac{Y(u)}{u} + 2Y(u) - \frac{y(u)}{u} + S[-2x(2x^2+3)y] = 0 \quad 22$$

$$\frac{d}{du}(uY(u) - y(u) + uS[-2(2x^3+3x)y] = 0$$

$$\frac{d}{du}(uY(u) - 1 = uS[2(2x^3+3x)y] \quad 23$$

Since S and S^{-1} represent the Sumudu Transform and Sumudu Inverse respectively, then the recursive relations can be obtained as;

$$y_0(x) = S^{-1}\left(\frac{1}{u}\overset{0}{\underset{u}{\int}} du\right)$$

$$S^{-1}\left(\frac{1}{u}(u)\right)$$

$$=1$$

$$y_{n+1}(x) = S^{-1}\left(\frac{1}{u}\int_0^u u\,(S[4x^3+6x]y_n(x)du)\right)$$

For n=0

$$y_1(x) = S^{-1}\left(\frac{1}{u}\int_0^u u\,(S[4x^3+6x]y_0(x)du)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\,(S[4x^3+6x]du)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\,(3!\,4u^3+6u)du\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\,(24u^4+6u^2)du\right)$$

$$= (x^3+3x)$$

$$= S^{-1}\left(\frac{1}{u}\left[\frac{24u^5}{5}+2u^3\right]\right)$$

$$= S^{-1}\left(\frac{1}{u}\left[\frac{24u^4}{5}+2u^2\right]\right)$$

$$y_1(x) = \frac{x^4}{5} + x^2$$

$$y_2(x) = S^{-1}\left(\frac{1}{u}\int_0^u u\,(S[4x^3+6x]y_1(x)du)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(S[4x^3+6x]\left(\frac{x^4}{5}+x^2\right)du\right)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(S\left[\frac{4x^7}{5}+4x^5+\frac{6x^5}{5}+6x^3\right]\right)du\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(4.7!\frac{u^7}{5}+4.5!\,u^5+5!\frac{6u^5}{5}+3!\,6u^3\right)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(7!\frac{4u^8}{5}+5!\,4u^6+4!\,6u^6+3!\,6u^4\right)\right)$$

$$= S^{-1}\left(\frac{1}{u}\left(7!\frac{4u^9}{45}+\frac{5!\,4u^7}{7}+\frac{4!\,6u^7}{7}+\frac{3!\,6u^5}{5}\right)\right)$$

$$= S^{-1}\left(7!\frac{4u^8}{45}+\frac{5!\,4u^6}{7}+\frac{4!\,6u^6}{7}+\frac{3!\,6u^4}{5}\right)$$

$$= \frac{x^8}{90}+\frac{13x^6}{105}+\frac{3x^4}{10}$$

$$y_3(x) = S^{-1}\left(\frac{1}{u}\int_0^u u\,(S[4x^3+6x]y_2(x)du)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(S[4x^3+6x]\left(\frac{x^8}{90}+\frac{3x^4}{10}+\frac{13x^6}{105}+\frac{x^8}{90}\right)du\right)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(S\left(\frac{12x^7}{10}+\frac{42x^9}{105}+\frac{4x^{11}}{90}+\frac{18x^5}{10}+\frac{78x^7}{105}+\frac{6x^9}{90}\right)du\right)\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(7!\frac{12u^7}{10}+9!\frac{42u^9}{105}+11!\frac{4u^{11}}{90}+5!\frac{18u^5}{10}+7!\frac{78u^7}{105}+9!\frac{6u^9}{90}\right)du\right)$$

$$= S^{-1}\left(\frac{1}{u}\int_0^u u\left(7!\frac{12u^8}{10}+9!\frac{42u^{10}}{105}+11!\frac{4u^{12}}{90}+5!\frac{18u^6}{10}+7!\frac{78u^8}{105}+9!\frac{6u^{10}}{90}\right)du\right)$$

$$= S^{-1}\left[\frac{1}{u}\left(7!\frac{12u^9}{90}+9!\frac{42u^{11}}{105x11}+11!\frac{4u^{13}}{90x13}+5!\frac{18u^7}{70}+7!\frac{78u^9}{105x9}+9!\frac{6u^{11}}{90x11}\right)\right]$$

$$= S^{-1}\left[7!\frac{12u^8}{90}+9!\frac{42u^{10}}{105x11}+11!\frac{4u^{12}}{90x13}+5!\frac{18u^6}{70}+7!\frac{78u8}{105x9}+9!\frac{6u^{10}}{90x11}\right]$$

$$y_3 = \frac{3x^6}{70}+\frac{17x^8}{630}+\frac{59x^{10}}{11550}+\frac{1x^{12}}{3510}$$

Therefore, the series solution is given as

$$y(x) = y_0 + y_1 + y_2 + y_3 + y_4 + \dots + y_n \qquad 24$$

Hence the solution

$y(x) = 1 + x + \frac{x^4}{2!} + \frac{1}{3!}x^6 + \frac{1}{4!}x^8 + ..$ 25

Equation (25) can be written in closed form as

$y(x) = e^{x^2}$ 26

Equation (26) is the exact solution of equation (19) which is the same as the result in Eltayeb *et al.* (2015).

Example 2

Consider the nonlinear Lame Emden Fowler equation (Olubanwo *et al*., 2015)

$\frac{d^2y}{dx^2} + \frac{2}{x}y' + \alpha(6y + 4y\,ln\,y) = 0$ 27

Let y(0)=1 , y'(0)=0

Multiply equation (27) by x and taking the Sumudu Transform yields,

$S\left[x\frac{d^2y}{dx^2} + 2y' + \alpha x(6y + 4y\,ln\,y)\right] = 0$ 28

Using the initial properties of Sumudu Transform, we obtain

$$\frac{dY}{du} - \frac{Y(u)}{u} + 2Y(u) - \frac{y(u)}{u} + S[(6y + 4y\,ln\,y)] = 0$$

$\frac{d}{du}(uY(u) - 1 = -uS[(6\alpha xy + 4\alpha xy\,ln\,y)] = 029$

Integrating equation (29) and taking the inverse Sumudu Transform leads to;

$y(t) = S^{-1}\left(\frac{1}{u}\int_0^u du\right) + S^{-1}[(6\alpha xy + 4\alpha xy\,ln\,y)] = 0$ 30

Applying Homotopy perturbation method

$\sum_{n=0}^{\infty} p^n y_n = 1 + pS^{-1}\left(-\frac{1}{u}\int_0^u u\,(S[6\alpha x \sum_{n-0}^{\infty} p^n\, y_n(n) + 4\alpha \sum_{n=0}^{\infty} p^n y_n\, ln \sum_{n=0}^{\infty} p^n y_n])du\right)$ 31

Equating the coefficients of the corresponding power of p

$p^0: y_0 = 1.$

$$p^1: y_1 = -S^{-1}\left(\frac{1}{u}\int_0^u u(S[6\alpha x y_0 + 4\alpha y_0 x\,ln\,y_0])du\right)$$

$$= S^{-1}(2\alpha u^2) = -\alpha x^2.$$

$$p^2: y_2 = -S^{-1}\left(\frac{1}{u}\int_0^u u(S[6\alpha x y_1 + 4\alpha y_1 x\,ln\,y_1])du\right)$$

$$= S^{-1}\left(\frac{10x3!}{5}.u^4\right) = \frac{1}{2!}x^4.$$

Similarly, we obtain

$$y_3 = -\frac{1}{3!}x^2$$

$$y_4 = -\frac{1}{4!}x^8.$$

Therefore, the series solution is given as

$y(x) = y_0 + y_1 + y_2 + y_3 + \cdots$ 32

$y(x) = 1 - \alpha x^2 + \frac{1}{2!}\alpha^2 x^4 - \frac{1}{3!}\alpha^3 x^6 + \frac{1}{4!}\alpha^4 x^8.$ 33

Equation (33) can be written in closed form as

$$y(x) = e^{-\alpha x^2} \qquad 34$$

Thus, the equation (34) is the exact solution of equation (27) which is in good agreement as the result obtained in Olubanwo *et al.* (2015).

Example 3

Consider the Lame Emden equations (Olayiwola and Adegoke, 2019)

$$\frac{d^2y}{dx^2}+\frac{2}{x}\left(y+\frac{dy}{dx}\right)-y=0 \qquad 35$$

Let $y(0)=1, y'(0)=0$

Multiplying equation (35) by x and taking the Sumudu Transform yields

$$S\left[x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+2y-xy\right]=0 \qquad 36$$

Applying the initial properties of Sumudu Transform

$$\frac{dY}{du}-\frac{Y(u)}{u}+2Y(u)-\frac{y(u)}{u}+S[y(2-x)]=0$$

$$\frac{d}{du}(uY(u)-1=-uS[(2-x)y]=0 \qquad 37$$

Integrating equation (37) and taking the inverse Sumudu Transform leads to

$$y(t)=S^{-1}\left(\frac{1}{u}\int_0^u du\right)+S^{-1}\left(\frac{1}{u}\int_0^u u(S[(2-x)y])du\right)=0 \qquad 38$$

Applying Homotopy perturbation method (HPM)

$$\sum_{n=0}^{\infty}p^n y_n = 1+pS^{-1}\left(-\frac{1}{u}\int_0^u u\,(S[(2-x)\sum_{n-0}^{\infty}p^n\,y_n(n)]du)\right) \qquad 39$$

Equating the coefficients of the corresponding power of p, then;

$p^0: y_0 = 1.$

$$p^1: y_1=-S^{-1}\left(\frac{1}{u}\int_0^u uS[(2-x)y_0]du\right)$$

$$p^1: y_1=-S^{-1}\left(\frac{1}{u}\int_0^u [(2u-u^2)]du\right)$$

$$=S^{-1}\left[\frac{1}{u}\left(\frac{2u^2}{2}-\frac{u^3}{3}\right)\right]$$

$$=S^{-1}\left(2u-\frac{u^2}{3}\right)$$

$$=-x+\frac{x^2}{3!}$$

$$p^2: y_2=-S^{-1}\left(\frac{1}{u}\int_0^u u[S(2-x)y_1]du\right)$$

$$y_2=-S^{-1}\left(\frac{1}{u}\int_0^u u\left[S\left(-2x+\frac{2x^2}{3!}+x^2-\frac{x^3}{3!}\right)\right]du\right)$$

$$y_2=-S^{-1}\left(\frac{1}{u}\int_0^u u\left[-2u+\frac{4u^2}{6}+2u^2-\frac{3!u^3}{3!}\right]du\right)$$

$$=-S^{-1}\left(\frac{1}{u}\int_0^u\left(-2u^2+\frac{2}{3}u^3+2u^3-u^4\right)du\right)$$

$$=-S^{-1}\left(\frac{2u^2}{3}-\frac{2u^3}{3}+\frac{u^4}{5}\right)$$

$$=\frac{x^2}{3}-\frac{x^3}{9}+\frac{x^4}{5!}$$

$$p^3: y_3=-S^{-1}\left(\frac{1}{u}\int_0^u u[S(2-x)y_2]du\right)$$

$$y_3=-\frac{1}{18}x^3+\frac{1}{36}x^4-\frac{23}{5400}x^5+\frac{1}{5040}x^6$$

$$y_4=-S^{-1}\left(\frac{1}{u}\int_0^u u[S(2-x)y_3]du\right)$$

$$y_3=-\frac{1}{180}x^4-\frac{1}{270}x^5+\frac{7}{8100}x^6+\frac{11}{132300}x^7-\frac{1}{362880}x^8$$

Therefore, the series solution is given as

$$y(x)=y_0+y_1+y_2+y_3+y_4+\ldots+y_n$$

$$y(x)=1-x+\tfrac{1}{2}x^2-\tfrac{1}{6}x^3+\tfrac{1}{24}x^4\ldots\ldots\ldots \qquad 40$$

Applying Taylor's series, Equation (40) can be written as

$$y(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!} \qquad 41$$

Equation (40) is the solution of equation (35) which is the same as the result in (Olayiwola and Adegoke, 2019).

Conclusion

In this paper, Sumudu Transform with Homotopy Perturbation Method was utilized to obtain the exact solutions Lane-Emden type singular initial value problems. The main advantage of Sumudu Transform with Homotopy Perturbation Method is that, it provides the solution in series of rapidly convergent sequence and introduces a significant advancement in solving singular initial value problems over existing methods. The solutions obtained by this method agreed with other solutions in the references.

Conflict of Interest

The authors declare that there is no conflict of interest related to this work.

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